9/3 Recitation Solutions

1. Consider the map $T : \mathbb{R}^2 \to \mathbb{R}^3$ given by $T\Big(\begin{bmatrix} x \\ y \end{bmatrix}\Big) = \begin{bmatrix} 2x+y \\ x \\ -2y \end{bmatrix}$.

   Verify that T is a linear transformation and write down the matrix associated to T.

   We need to verify the two linear transformation properties:

   - $T\Big(\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} + \begin{bmatrix} x_1 \\ y_1 \end{bmatrix}\Big) = T\Big(\begin{bmatrix} x_1+x_2 \\ y_1+y_2 \end{bmatrix}\Big) = \begin{bmatrix} 2(x_1+x_2)+(y_1+y_2) \\ (x_1+x_2) \\ -2(y_1+y_2) \end{bmatrix}$
   $= \begin{bmatrix} 2x_1+y_1+2x_2+y_2 \\ x_1+x_2 \\ -2y_1-2y_2 \end{bmatrix}$
   $= \begin{bmatrix} 2x_1+y_1 \\ x_1 \\ -2y_1 \end{bmatrix} + \begin{bmatrix} 2x_2+y_2 \\ x_2 \\ -2y_2 \end{bmatrix}$
   $= T\Big(\begin{bmatrix} x_1 \\ y_1 \end{bmatrix}\Big) + T\Big(\begin{bmatrix} x_1 \\ y_1 \end{bmatrix}\Big)$

   - $T\Big(c\begin{bmatrix} x \\ y \end{bmatrix}\Big) = T\Big(\begin{bmatrix} cx \\ cy \end{bmatrix}\Big)$ $\begin{bmatrix} 2(cx)+(cy) \\ cx \\ -2(cy) \end{bmatrix} = \begin{bmatrix} c(2x+y) \\ cx \\ c(-2y) \end{bmatrix} = cT\Big(\begin{bmatrix} x \\ y \end{bmatrix}\Big)$

   The matrix associated to T is $\begin{bmatrix} 2 & 1 \\ 1 & 0 \\ 0 & -2 \end{bmatrix}$.

2. In this problem, we will verify (in an example) that the operation of matrix multiplication corresponds to function composition (as discussed in lecture). Let $S, T : \mathbb{R}^2 \to \mathbb{R}^2$ be defined by

   $$S\Big(\begin{bmatrix} x \\ y \end{bmatrix}\Big) = \begin{bmatrix} 2x+y \\ x \end{bmatrix} \text{ and } T\Big(\begin{bmatrix} x \\ y \end{bmatrix}\Big) = \begin{bmatrix} x-y \\ y \end{bmatrix}$$

   Let A be the matrix encoding S and let B be the matrix encoding T.

   (a) Determine A and B.

   $$A = \begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix} \text{ and } B = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$$

(b) Compute AB.

$$AB = \begin{bmatrix} 2 & -1 \\ 1 & -1 \end{bmatrix}.$$

(c) Next compute $(S \circ T)\Big(\begin{bmatrix} x \\ y \end{bmatrix} \Big)$ and the matrix encoding $S \circ T$. Finally, verify that this is the same matrix as AB.

$$(S \circ T)\Big(\begin{bmatrix} x \\ y \end{bmatrix} \Big) = S\Big(T\Big(\begin{bmatrix} x \\ y \end{bmatrix} \Big)\Big) = S\Big(\begin{bmatrix} x-y \\ y \end{bmatrix} \Big) = \begin{bmatrix} 2(x-y)+y \\ x-y \end{bmatrix} = \begin{bmatrix} 2x-y \\ x-y \end{bmatrix}.$$

Hence $S \circ T$ is encoded by the matrix $\begin{bmatrix} 2 & -1 \\ 1 & -1 \end{bmatrix}$, which is simply AB.

3. Use Julia to plot each of the systems below and determine (geometrically), which system has no solution. For each system with a unique solution, use Julia to solve the system.

a) $2x - 3y = 1$
$x + y = 2$

b) $2x - 3y = 1$
$4x - 6y = 2$

c) $2x - 3y = 1$
$x - \frac{3}{2}y = 2$

System a

Plot Code:

```
using Plots
f(x) = 2/3*x - 1/3
g(x) = -x+2
x = -5 : 0.01 : 5
plot(x,[f,g])
```

This system has a unique solution since the lines intersect in a single point. The solution is $x = 1.4, y = .6$, which is given by:

Code:

```
A = [2 -3 ; 1 1];
b = [1 ; 2];
x = A\b
```

System b

Plot Code:

```
using Plots
f(x) = 2/3*x - 1/3
```

```
g(x) = 2/3*x - 1/3
x = -5 : 0.01 : 5
plot(x,[f,g])
```

This system has infinitely many solutions since the lines are the same.

System c

Plot Code:

```
using Plots
f(x) = 2/3*x - 1/3
g(x) = 2/3*x - 2
x = -5 : 0.01 : 5
plot(x,[f,g])
```

This system has no solution since the associated graphs are parallel.

4. In question 3, you thought about systems with two variables. Geometrically, these correspond to lines in $\mathbb{R}^2$. Now let's think about systems of equations involving 3 variables; these correspond to planes in $\mathbb{R}^3$.

   (a) Visit `https://www.geogebra.org/m/zrsx8gwe`. Here you will see the picture associated to the system

   $$\begin{aligned} x + y + z &= 1 \\ x - y + z &= 1 \end{aligned}.$$

   Rotate the picture to get a sense of what is going on. What do the solutions to this system look like geometrically? How many solutions are there? Now change the equations to see other possibilities. What do the systems

   $$\begin{aligned} x + y + z &= 1 \\ -2x - 2y - 2z &= -2 \end{aligned} \quad \text{and} \quad \begin{aligned} x + y + z &= 1 \\ x + y + z &= 2 \end{aligned}$$

   look like? How many solutions do they have?

   The two planes given by the first system intersect in a line, giving infintely many solutions. The two planes in the second system are identical and hence intersect in all points on the plane (so there is once again infinitely many solutions). The two planes in the third system are parallel and hence there is no solution.

(b) Visit `https://www.geogebra.org/m/hb2zskvf`. Here you will see the picture associated to the system

$$\begin{aligned} x+y+z&=1\\ x-y+z&=1.\\ x+y-z&=1 \end{aligned}$$

How many solutions does this system have? What is it, geometrically? Next consider the systems

$$\begin{aligned} x+y+z&=1\\ x-y+z&=1\\ -x+y-z&=-1 \end{aligned} \quad \text{and} \quad \begin{aligned} x+y+z&=1\\ x-y+z&=1\\ x-y+z&=2 \end{aligned} \quad \text{and} \quad \begin{aligned} x+y&=1\\ x-y&=1.\\ -2x+y&=1 \end{aligned}$$

How many solutions do these systems have?

The three planes given by the first system intersect in a point, giving a unique solution. Two of the planes in the second system are identical and intersect the thrid plane in a line; hence all points on the line are solutions to the system. Two of the planes in the third system are parallel and hence there is no solution since there is no point in which all three planes intersect. The three planes in the fourth system do not intersect in any common points either; hence there is no solution.